Module length

Let M be an R-module. Instead of looking at increasing chains of submodules, like w/ Noetherian modules, we consider decreasing chains.

Def: M is Artinian if every strictly decreasing chain of submodules terminates (or ideals in the case of rings).

We'll soon see that all Artinian rings are always Noetherian. In fact, we'll prove something much stronger. First some definitions:

$M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_n$ is a chain of submodules of length n.

The chain is a composition series if each M_j / M_{j+1} is a nonzero simple module (i.e. it has no nonzero proper submodules).

Equivalently, a composition series is a max'l chain of submodules.

The length of M is the least length of a composition series, or ∞ if it has no finite composition series. (In fact, we'll see that all composition series have the same length.)

Remark: Since M_i/M_{i+1} is simple, it is generated by any nonzero element a. So, we get

$$\begin{array}{c} 1 \longmapsto a \\ R \twoheadrightarrow M_i/M_{i+1} \end{array} \quad \text{with kernel}$$

$$P = \{r \in R \mid r(M_i/M_{i+1}) = 0\} = \operatorname{ann}(M_i/M_{i+1}), \text{ so}$$

$M_i/M_{i+1} \cong R/P$. Since the module is simple, P must be maximal.

Thm: M has a finite composition series iff M is Artinian and Noetherian.

Pf: Suppose M is Artinian and Noetherian. By ACC, we can find a max'l proper submodule M_1, a max'l submodule M_2 of M_1 etc. By DCC, this terminates in finitely many steps.

For the other direction, we assert the following:

Claim: If M has a finite composition series of length n, then every chain of submodules has length $\leq n$ and can be refined to a composition series.

Sketch of Pf of Claim: Let $M' \subsetneq M$ a proper submodule. Consider $M' = M_0 \cap M' \supseteq M_1 \cap M' \ldots \supseteq M_n \cap M' = 0$.

Can show by induction that if these are all proper containments, $M' \supsetneq M_0 = M$, a contradiction.

Now, suppose $M = N_0 \supsetneq N_1 \cdots \supsetneq N_k$ is another chain of submodules. WTS $k \leq n$.

If $n=0$, then $M=0$, and we're done. By above, length $N_1 <$ length M, so by induction, $k-1 \leq n-1$, and we're done. □

The claim finishes the proof of the theorem since every chain has finite length. □

Now we come back to rings to see how we can ultimately interpret Artinianness geometrically.

Thm: Let R be a ring. The following are equivalent.

a.) R is Noetherian and all its prime ideals are maximal.

b.) R is a finite length R-module.

c.) R is Artinian.

Moreover, if these are satisfied, R has only finitely many maximal ideals.

Pf: a.) $\Rightarrow$ b.) Suppose R is not of finite length.

Let $I \subset R$ be an ideal max'l w.r.t. the property that R/I is not of finite length.

WTS I is prime. Let $a, b \in R$ s.t. $ab \in I$. Consider $I + (a) \supseteq I$. If $a \notin I$ this containment is proper, so $R/_{I+(a)}$ has finite length.

We have an exact sequence

$$0 \to \frac{I+(a)}{I} \longrightarrow R/I \longrightarrow R/_{I+(a)} \to 0$$

But notice: $\frac{R}{(I:a)} \xrightarrow{\cdot a} \frac{I+(a)}{I}$ is an isomorphism!

(Surjectivity is clear, and if $x \mapsto 0$ then $xa \in I$ so $x \in (I:a)$)

If $b \notin I$, then $(I:a) \supsetneq (I)$, so $R/_{(I:a)}$ also has finite length.

Combining the two composition series, R/I has finite length, which is a contradiction.

Thus I is prime, and thus max'l, so R/I is a field. Thus R does in fact have finite length.

b.) $\Rightarrow$ c.) Follows from previous theorem.

c.) $\Rightarrow$ a.) Suppose R is Artinian.

Claim: (0) is a product of max'l ideals.

Pf of Claim: Since R is Artinian, we can choose a min'l ideal J among products of max'l ideals.

Thus $\forall$ max'l $m \subsetneq R$, $mJ = J \Rightarrow J \subseteq m$, and $J^2 = J$.

Choose I min'l among ideals not annihilating J.

Then $(IJ)J = IJ^2 = IJ \neq 0$.

But $IJ \subseteq I$ so by minimality $IJ = I$.

Choose $f \in I$ s.t. $fJ \neq 0$. By minimality, $(f) = I$.

Since $IJ = I$, $\exists\, g \in J$ s.t. $f = fg \Rightarrow f(1-g) = 0$.

But g is in every max'l ideal, so $1-g$ is in none, so $1-g$ is a unit. $\Rightarrow f = 0. \Rightarrow J = 0.$ $\square$

Thus, we now have $0 = m_1 \cdots m_t$ for m_i max'l ideals, and $\forall\, s$, $m_1 \cdots m_s / m_1 \cdots m_{s+1}$ is a vector space over R/m_{s+1}.

Any descending chain of subspaces corresponds to a chain of ideals in R, which is finite. Thus, the vector space is finite dimensional.

Putting together these composition series, we get a finite composition series for R, so R has finite length, so it's Noetherian.

Now, suppose P is prime. Then $P \supseteq 0 = m_1 \dots m_t$
$\Rightarrow P \supseteq m_i$, some i. $\Rightarrow P$ is max'l.

In particular, every max'l ideal is one of the m_i, so there are only finitely many. □

What does this mean geometrically?

Cor: Let R be a ring. Then R is Artinian $\Rightarrow$ Spec(R) is finite. (In fact, the converse holds as well.)

In fact, if R is a k-algebra, with $k = \bar{k}$ (e.g. $\frac{k[x_1,\dots,x_n]}{I}$), then in this case $R \cong k^\ell$ as k-vector spaces, and ℓ = length of R = # of pts of Spec(R) (up to multiplicity)

Ex: 1.) $R = \frac{k[x,y]}{(x,y)}$. Spec(R) = $\{(0)\}$ and has length 1 : $R \supseteq (0)$

Also, $\frac{k[x,y]}{(x,y)} \cong k$.

2.) Let $R = \frac{k[x]}{x(x-1)}$. $\operatorname{Spec}(R) = \{ \underset{\text{point at 0}}{(x)}, \underset{\text{point at 1}}{(x-1)} \}$ (0) isn't prime!

R is generated as a k-vector space by 1 and x,

so $R \cong k^2$, and it has comp. series $R \supsetneq (x) \supsetneq (0)$.

3.) Let $R = \frac{k[x,y]}{(x, y^2)}$. $\operatorname{Spec}(R) = \{(x,y)\}$.

But $R \cong k^2$ (gen. by $1, y$), and it has comp. series

$R \supsetneq (y) \supsetneq (0)$.

Spec(R) corresponds to a "scheme-y" point:

$\uparrow$ a point w/ a tangent direction.

Think of this as roughly the limit of two colliding points on a line.